PHYSICAL REVIEW E, VOLUME 63, 055203

Loss of synchronization in coupled oscillators with ubiquitous local stability

Ned J. Corron*
U.S. Army Aviation and Missile Command, AMSAM-RD-WS-ST, Redstone Arsenal, Alabama 35898
(Received 6 December 2000; published 23 April 2001)

The issue of using instantaneous eigenvalues as indicators of synchronization quality in coupled chaotic systems is examined. Previously, it has been assumed that, if the eigenvalues of the linearized synchronization dynamics have negative real parts everywhere on the attractor, the synchronized state is stable. In this Rapid Communication, two counterexamples are presented that show this assumption is invalid.

DOI: 10.1103/PhysRevE.63.055203 PACS number(s): 05.45.Xt

The determination of necessary and sufficient conditions for high-quality synchronization of chaotic systems is currently an open question and the subject of much discussion. This question is of growing importance as chaotic waveforms emerge as possible candidates for spread-spectrum communication or radar waveforms. In this Rapid Communication, we address the issue of using instantaneous eigenvalues as indicators of synchronization quality and show, with two counterexamples, that an assumption that has been cited for assuring high-quality synchronization is invalid.

For two chaotic oscillators with unidirectional coupling, determining the stability of a synchronized state usually requires calculating the Lyapunov exponents for the response system, and synchronization is expected when all the exponents are negative [1]. However, it is well known that this condition does not always assure high-quality synchronization due to localized instabilities within the attractor [2,3]. Consequently, various stronger constraints have been proposed that incorporate the local synchronization dynamics [4–9]. Among these constraints, it has been tacitly postulated that a sufficient condition for high-quality synchronization can be expressed using the instantaneous eigenvalues of the driven response system [5,9]. Explicitly, it is assumed that if the eigenvalues of the linearized synchronization dynamics have negative real parts everywhere on the attractor, the synchronized state is stable. Intuitively this makes sense: the local dynamics appear everywhere contracting. Indeed, this criterion was employed with great success to optimize the choice of scalar coupling and maintain synchronization between chaotic systems with large parameter mismatch [5].

However, the use of instantaneous eigenvalues to determine the stability of time-varying systems can be misleading. For example, we consider the simple linear oscillator with time-varying coefficients

$$\ddot{\xi}+0.1\dot{\xi}+(1+0.9\cos t)\xi=0, \tag{1}$$

where the dot denotes differentiation with respect to time t. Equation (1) is a particular case of Mathieu's equation with damping [10]. Although this linear oscillator exhibits positive damping, the equilibrium state $\xi=0$ is actually unstable due to a resonance excited by the parametric modulation. In constructing the two counterexamples shown below, we use this unstable oscillator as a model for the linearized synchronization dynamics.

Interpretation of the instability in Eq. (1) in terms of the system's instantaneous eigenvalues and eigenvectors is not intuitive; however, that negative eigenvalues can allow growth in a linear system is not surprising. Indeed, it is known that non-normal, constant-coefficient linear systems with negative eigenvalues can exhibit an initial growth before decaying exponentially [9]. Geometrically, this transient growth is due to nonorthogonal eigenvectors with eigenvalues indicating different decay rates. For certain initial conditions, motion parallel to the eigenvector corresponding to the fast decay rate carries the system state away from the equilibrium point before the dynamics along the slow eigenvector eventually bring the state back to equilibrium. In a time-varying linear system such as Eq. (1), one anticipates that such transient growth can be sustained by the changing eigenvalues and eigenvectors, thereby providing a possible mechanism for instability.

To investigate synchronization, we consider two identical chaotic oscillators with unidirectional coupling. The drive system is

$$\dot{u}=f(u), \tag{2}$$

where $u=u(t)$ is a vector of drive system states and f is a nonlinear vector function defining the flow. For the following, it is assumed that system (2) exhibits chaotic dynamics. The response system is

$$\dot{v}=f(v)+[g(u)-g(v)], \tag{3}$$

where $v=v(t)$ is a vector of response system states and g is a vector coupling function, possibly nonlinear. Loosely, $g(u)$ can be identified as the signal transmitted from the drive system to the response system.

For any trajectory $u(t)$ generated by drive system (2), the synchronization state

$$v(t)=u(t) \tag{4}$$

is a solution of response system (3). To investigate the stability of this state, we define

$$\varepsilon=v-u, \tag{5}$$

*Electronic address: ned.corron@ws.redstone.army.mil

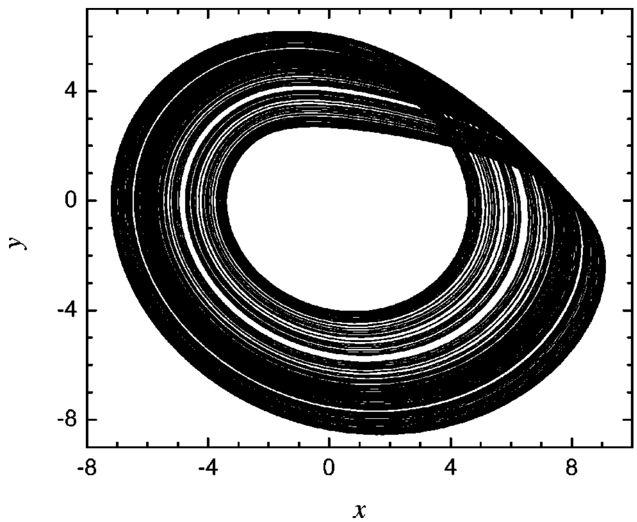

FIG. 1. Phase space projection of the Rossler attractor generated by flow (9) and showing $x > -8$.

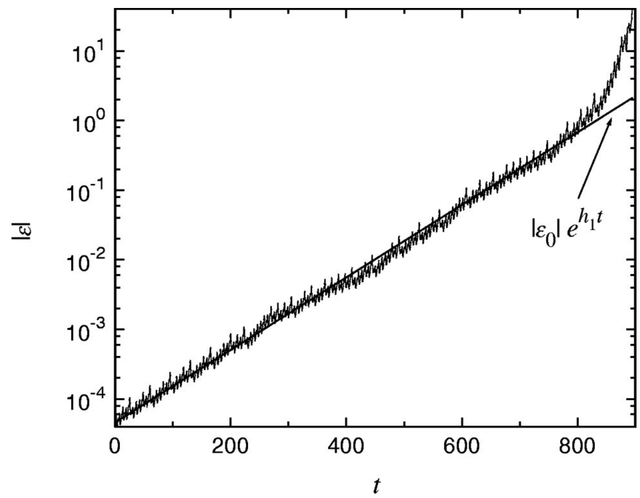

FIG. 2. Loss of synchronization for the Rossler systems with coupling (10), including the expected exponential divergence from the initial deviation ε_0 due to the single positive Lyapunov exponent $h_1 = 0.012$.

which evolves as

$$\dot{\varepsilon} = [f(v) - f(u)] - [g(v) - g(u)]. \tag{6}$$

For small perturbations from the synchronization state, the dynamics can be approximated as

$$\dot{\varepsilon} = J\varepsilon, \tag{7}$$

where

$$J = Df(u) - Dg(u) \tag{8}$$

and $Df(u)$ and $Dg(u)$ contain the partial derivatives of the functions f and g with respect to their arguments and evaluated along the drive trajectory $u(t)$. In general, coefficient matrix (8) is time varying and the stability of the synchronization state cannot be immediately inferred. To assure high-quality synchronization, it has been assumed that a sufficient condition is that all the instantaneous eigenvalues of matrix (8) have negative real parts everywhere on the attractor [5,9]. However, we present two counterexamples to show this assumption is invalid.

For the first counterexample, we consider two coupled Rossler oscillators of the forms (2) and (3), where the flow is

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-z \\ x+0.2y \\ 0.2+z(x-4.5) \end{pmatrix} \tag{9}$$

and the coupling function is

$$g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0.3(y-x)-0.08125x^2 \\ y+z(x-3.5) \end{pmatrix}. \tag{10}$$

The simply folded band attractor generated by flow (9) is shown in Fig. 1. The corresponding coefficient matrix (8) is

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1.3+0.1625x & -0.1 & 0 \\ 0 & -1 & -1 \end{pmatrix}. \tag{11}$$

The instantaneous eigenvalues for this matrix are -1 and $-0.05 \pm \sqrt{-1.2975-0.1625x}$. As seen in Fig. 1, the attractor is characterized by $x > -8$; thus, the eigenvalues have negative real parts everywhere on the attractor, which suggests the synchronization state is stable. However, numerical integration of the drive and response systems shows that high-quality synchronization is not attained. A typical result of numerically integrating the system defined by Eqs. (2), (3), (9), and (10) is shown in Fig. 2 [11]. In this example, a small initial deviation ε_0 between the drive and response systems is introduced at $t = 0$. As the coupled systems evolve, the magnitude of the deviation (calculated using the standard Euclidean norm) grows exponentially, indicating a divergence of the drive and response systems due to unstable linear synchronization dynamics. Ultimately, the deviation grows to the size of the attractor, where nonlinear effects become important and the rate of divergence is affected; at this point, synchronization is lost. In fact, for the coupled Rossler systems, the response system is often driven from the chaotic attractor (with $z < 0$) and grows unbounded. For example, an initial departure from the attractor is evident in Fig. 2 at $t > 800$.

It is important to note that the fundamental instability in this example is not due to nonlinear effects: the synchronization state is linearly unstable. To show this, we examine linear system (7) with time-varying coefficient matrix (11). Writing $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3)^T$, we immediately see that the $\varepsilon_1 - \varepsilon_2$ subsystem decouples to give

$$\ddot{\eta} + 0.1\dot{\eta} + (1.3 + 0.1625x)\eta = 0, \tag{12}$$

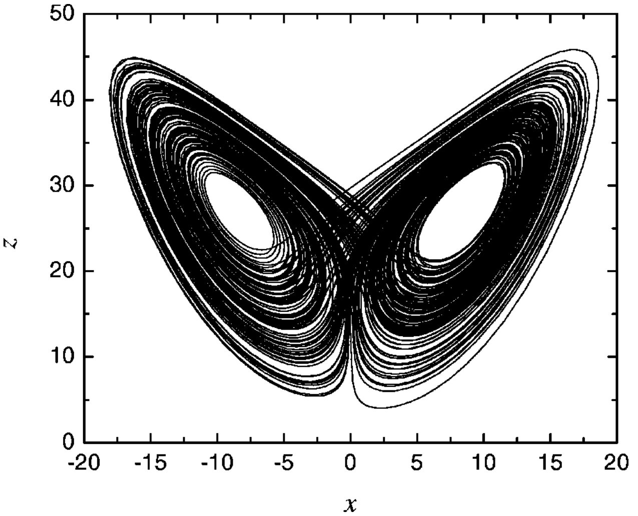

FIG. 3. Phase space projection of the Lorenz attractor generated by flow (13) and showing $z>0$.

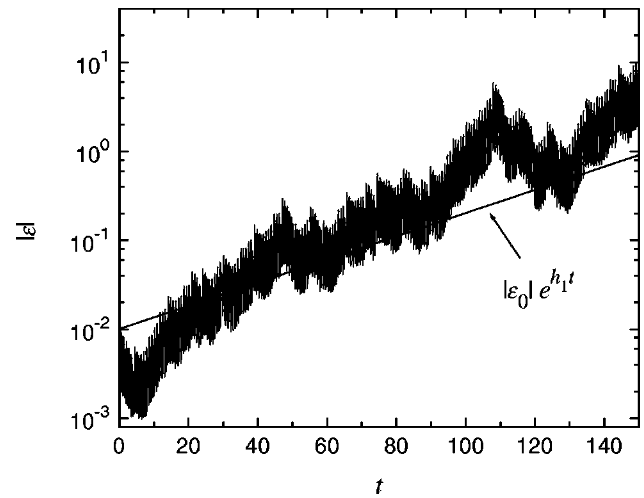

FIG. 4. Loss of synchronization for the Lorenz systems with coupling (14), including the expected divergence from the initial deviation ε_0 due to the single positive Lyapunov exponent $h_1 = 0.03$.

where $\varepsilon_1 = \eta$ and $\varepsilon_2 = -\dot{\eta}$. (For completeness, ε_3 is driven as $\dot{\varepsilon}_3 = \dot{\eta} - \varepsilon_3$.) Equation (12) is similar to unstable linear oscillator (1), except that the periodic coefficient in (1) is now replaced by the chaotic drive x. Although x is not periodic, it does exhibit a strong spectral component due to Rossler's simply folded band attractor, and it is reasonable to expect that Eq. (12) may exhibit instability due to parametric resonance. Indeed, numerical integration shows that solutions to Eq. (12) grow unbounded, implying that the linear synchronization dynamics are unstable. Furthermore, calculations of the Lyapunov exponents for the response system (using a standard numerical technique [12]) yield $h_1 = 0.012$, $h_2 = -0.112$, and $h_3 = -1.000$. Since the Lyapunov exponents are derived from the linearized synchronization dynamics, the existence of a positive exponent confirms that the synchronization state for the coupled Rossler systems is linearly unstable. As seen in Fig. 2, the rate of divergence of the coupled oscillators is consistent with the linear instability predicted by $h_1 > 0$.

For the second example, we consider two coupled Lorenz oscillators, with

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10(y-x) \\ x(28-z)-y \\ xy-2.6667z \end{pmatrix} \tag{13}$$

and

$$g\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10y-9.95x-z^2 \\ x(27-z) \\ x(y+1.445)-2.6167z \end{pmatrix}. \tag{14}$$

The familiar Lorenz attractor generated by flow (13) is shown in Fig. 3. For the coupled system, coefficient matrix (8) is

$$J = \begin{pmatrix} -0.05 & 0 & 2z \\ 1 & -1 & 0 \\ -1.445 & 0 & -0.05 \end{pmatrix} \tag{15}$$

and the instantaneous eigenvalues are -1 and $-0.05 \pm \sqrt{-2.89z}$. As seen in Fig. 3, the Lorenz attractor has $z > 0$; thus, the eigenvalues have negative real parts everywhere on the attractor. For this example, we designed the $\varepsilon_1 - \varepsilon_3$ subsystem of linear system (7) with coefficient matrix (15) to mimic unstable oscillator (1). Shown in Fig. 4, numerical calculations of the coupled Lorenz systems indicate that small perturbations from the synchronization state grow with time, ultimately growing to the size of the attractor and synchronization is lost [13]. Calculations of the response system Lyapunov exponents yield $h_1 = 0.03$, $h_2 = -0.18$, and $h_3 = -0.95$, which confirm that the synchronization state for the coupled Lorenz oscillators is linearly unstable.

Although these counterexamples demonstrate that the instantaneous eigenvalues of J are insufficient for assuring stable synchronization, a sufficient condition based only on the instantaneous eigenvalues of the symmetric matrix $J + J^T$ has been developed [3]. It is, if all eigenvalues of $J + J^T$ are negative everywhere on the attractor, the synchronization dynamics are necessarily stable. The sufficiency of this condition can be easily proven using straightforward techniques. For the two counterexamples presented here, $J + J^T$ is characterized by a positive eigenvalue throughout each attractor; thus, this stronger condition is clearly not met and stability is not assured for either example. Unfortunately, this condition appears to be overly strong for many cases, as couplings can be found that do not meet this requirement yet still provide a stable synchronization state.

In conclusion, we have shown by counterexample that requiring the instantaneous eigenvalues for the synchronization dynamics to have negative real parts everywhere on the attractor is insufficient for assuring high-quality synchronization of coupled chaotic systems. However, we acknowledge that these counterexamples employ contrived couplings, chosen specifically to excite a resonant instability. For many practical systems with more conventional coupling, this requirement may provide effective results [5].

The author wishes to acknowledge Dan Hahs for helping to identify this problem and Jonathan Blakely, Shawn Pethel, Charles Bowden, and Krishna Myneni for many valuable comments and suggestions.

[1] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).

[2] P. Ashwin, J. Buescu, and I. Stewart, Phys. Lett. A **193**, 126 (1994).

[3] D. J. Gauthier and J. C. Bienfang, Phys. Rev. Lett. **77**, 1751 (1996).

[4] R. Brown and N. F. Rulkov, Phys. Rev. Lett. **78**, 4189 (1997).

[5] G. A. Johnson, D. J. Mar, T. L. Carroll, and L. M. Pecora, Phys. Rev. Lett. **80**, 3956 (1998).

[6] Z. Galias, Int. J. Circ. Theory. Appl. **27**, 589 (1999).

[7] F. Ali and M. Menzinger, Chaos **9**, 348 (1999).

[8] Z. Zhu, H. Leung, and Z. Ding, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **46**, 1320 (1999).

[9] J. N. Blakely, D. Gauthier, G. Johnson, T. L. Carroll, and L. M. Pecora, Chaos **10**, 738 (2000).

[10] D. W. Jordan and P. Smith, *Nonlinear Ordinary Differential Equations* (Oxford University Press, Oxford, 1977), Sect. 8.7.

[11] For all calculations, we use MATLAB (version 5.3) and its ODE45 solver, which is a Runge-Kutta (4,5) algorithm with an adjustable step size. The maximum step size used for integrating the coupled Rossler systems is $\Delta t = 0.585$.

[12] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, England, 1993), Sect. 4.4.

[13] The maximum step size used for integrating the coupled Lorenz systems is $\Delta t = 0.0835$.